Vergence, vision, and geometric optics

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The vergence treatment of geometric optics provides both conceptual and algebraic advantages over the traditional object distance–image distance–focal length formalism. These advantages are particularly useful for the life science student. In this paper vergence is defined in terms of the curvature of the wave fronts, and some examples are given to illustrate the advantages of this approach.

I. INTRODUCTION

The teaching of physics to life science students has generated considerable interest in the past three or four years. However as manifested in texts which have appeared,[1,2] geometric optics continues to be treated in the traditional manner. The basic equation used is the thin lens equation connecting the image distance v, the object distance u, and the posterior focal length f_2. If the positive direction is taken to be the direction in which the light is traveling, and all distances are measured from the lens to the position being considered, then the equation may be written as

$$1/v = 1/f_2 + 1/u. \tag{1}$$

For many life science students, the solution of Eq. (1) is a formidable task involving common denominators and the manipulation of fractions. Even for those who have no difficulty with the algebra, the task is still tedious. In addition, conceptual difficulties pop up in the cases of diverging light leaving a lens (virtual images), and converging light incident on a lens (virtual objects). The latter situation is avoided even by some calculus level texts for physics majors.[3] However, these two cases are everyday occurrences in the correction of myopia (nearsightedness) and hyperopia (farsightedness). The concept of vergence, used by optometrists and ophthalmologists in their treatment of geometric optics, offers both algebraic and conceptual advantages over the traditional physics approach [Eq. (1)]. The purpose of this paper is to make the physics community aware of these advantages.

Vergence is usually defined in terms of rays,[4–8] but it can also be defined in terms of the curvature of the wavefronts.[9] I will use the wave-front approach since it is clearer and easier to grasp than the ray approach. The remainder of the paper is divided into three sections. Section II deals with the combined wave front–vergence approach to thin lenses in air (with the assumption that $n = 1$, where n is the index of refraction of the air). Section III deals with the generalization of the vergence approach to light propagation in media other than air (i.e., $n \neq 1$). Section IV concludes the article.

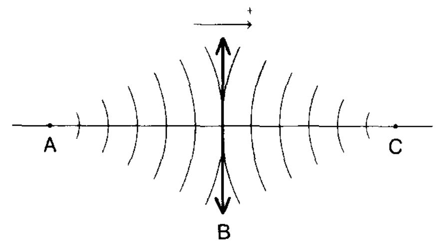

Fig. 1. Light waves diverging from a point source at A, passing through a convex thin lens at B, and converging to form a real image at C.

II. VERGENCE—THIN LENSES IN AIR ($n = 1$)

Let us first consider how a lens works, using some intuitive ideas about waves. An ideal point source will emit spherical diverging light waves. Given the usual first-order approximations (i.e., aberrations and diffraction neglected), the waves leaving a thin lens are also spherical but the emerging wave front has a different curvature than the incident wave front. In the case shown in Fig. 1, the converging waves form a point image conjugate to the point object.

For light in air ($n = 1$), vergence at a given position is defined as the curvature of the wave front at that position. Vergence is a measure of the degree of convergence or divergence of the light at the given position. In the case being considered, the wave fronts are spherical, and the curvature of a sphere is equal to the reciprocal of the radius of the sphere.[10] The unit of vergence in universal use is the diopter (abbreviated D) which is dimensionally equal to the reciprocal of a meter. I will take diverging light as having negative vergence, and converging light as having positive vergence. Historically, concepts similar to vergence date back to the British astronomer J. F. W. Herschel in 1827, while the diopter was introduced by the French ophthalmologist Ferdinand Monoyer in 1872.[11]

From Fig. 1, one expects that vergence is large (in magnitude) near a point source or point image (the wave fronts are highly curved), and small far from a point source or point image (the wave fronts are fairly flat). At 0.25 m from the point source, the vergence V of the diverging light is -4 D; at 0.5 m, $V = -2$ D (not as divergent); at 1 m, $V = -1$ D; at 2 m, $V = -0.5$ D (even closer to becoming plane). Converging light of vergence $+4$ D at a position means the radius of the wave front at that position is 0.25 m. If allowed to continue on, the converging light will form a point image 0.25 m away. The vergence of the converging light 5 cm from the image position is $(100\ \text{cm/m})(5\ \text{cm})^{-1} = +20$ D.

In terms of the vergence U of the light incident on a lens and the vergence V of the light as it leaves the lens, the thin lens equation is

$$V = P + U, \tag{2}$$

where P is the dioptric power of the lens. The dioptric power is a measure of the ability of the lens to converge or diverge light. If light of vergence $U = -2$ D is incident and light of vergence $V = -5$ D is leaving, then $P = -3$ D. Because of the simplicity of Eq. (2) and the resulting formalism, ophthalmic lenses are marked in terms of dioptric power.

As an example of an imaging problem, let us imagine that an illuminated Van Gogh painting is located 25 cm in front of a +7-D lens. Diverging light from the painting reaches the lens with a vergence $U = -100/25 = -4$ D. From Eq. (2), $V = 7 + (-4) = +3$ D. The light leaving the lens is converging, and the wave front has a radius of $v = 0.33$ m $= 33.3$ cm. If uninterrupted, this light will form a real image 33.3 cm behind the lens.

From the above example, it is clear that Eq. (2) is algebraically much easier to work with than Eq. (1). The operations required consist only of divisions, and an addition or subtraction, whereas Eq. (1) is usually solved with fractions. Furthermore, Eq. (2) has the conceptual advantage of describing the process in terms of what the light is actually doing at the lens as opposed to Eq. (1), which describes the process in terms of some distant object or image. Treated in this manner, a converging wave front incident on a lens (a virtual object) is just as easy to understand as a diverging wave front incident on a lens (a real object). Similarly, a diverging wave front leaving a lens (a virtual image) is just as easy to understand as a converging wave front leaving a lens (a real image).

It is obvious that the object distance u is the radius of curvature of the incident wave front, and the image distance v is the radius of curvature of the outgoing wave front. With the sign conventions mentioned above,

$$u = 1/U \quad \text{and} \quad v = 1/V. \tag{3}$$

For plane waves incident on a lens, $U = 0$ and, from Eq. (2), $V = P$. The image distance in this case is defined to be the posterior focal length f_2, and it follows from Eq. (3) that

$$f_2 = 1/P. \tag{4}$$

The posterior focal length of the above +7-D lens is $f_2 = (100 \text{ cm/m})(7 \text{ D})^{-1} = +14.3$ cm. For a -3-D lens, $f_2 = -33.3$ cm.

For plane waves leaving a lens, $V = 0$ and, from Eq. (2), $U = -P$. The object distance in this case is defined to be the anterior focal length f_1, and it follows that

$$f_1 = -1/P. \tag{5}$$

For the +7- and -3-D lenses, $f_1 = -14.3$ and $+33.3$ cm, respectively.

When two thin lenses (dioptric power P_1 and P_2, respectively) are placed in contact with each other, then the light leaving the first lens is immediately incident on the second lens. Therefore, $V_1 = U_2$, where V_1 is the vergence of the light leaving the first lens and U_2 is the vergence of the light incident on the second lens. It is then easy to show that the total dioptric power P of the two lens system is

$$P = P_1 + P_2. \tag{6}$$

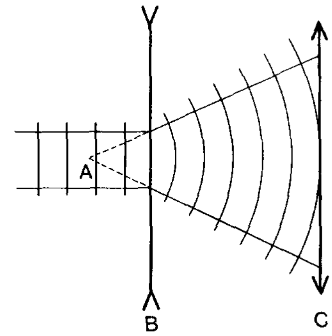

Fig. 2. Diverging wave fronts moving from a concave (minus) lens at B toward the convex (plus) lens at C. The center of curvature of each of the wave fronts is at A.

When the two thin lenses have an air gap between them, the situation is different. Figure 2 shows a diverging (minus) lens B separated by a certain distance from a converging (plus) lens C. Plane waves are incident on lens B, and the waves leaving lens B are diverging and have their center of curvature at point A. As the waves move from lens B toward lens C, they become flatter (less curved) and hence the vergence of the light incident on lens C is less than the vergence of the light leaving lens B. As an example, suppose lens B has dioptric power -11.6 D, and the distance between B and C is 1.38 cm. Point A (the center of curvature of the wave fronts leaving lens B) is located a distance $v = 100/(-11.6) = -8.62$ cm from lens B. Lens C is a distance of 10.00 cm (8.62 + 1.38 cm) from point A, or the radius of curvature of the wave front incident on lens C (the object distance) is $u = -10.00$ cm. The vergence of the light incident on lens C is then $U = 100/(-10) = -10.00$ D. Note that for converging light leaving a lens the vergence increases as the waves move away from the lens. These are the standard "lens effectivity" results.

The above example can be very nicely applied to explain the differences in dioptric power between a spectacle lens corrections and a contact lens correction. In Fig. 2, lens C could represent the cornea of a myopic eye, and lens B is then the spectacle lens correction. The vergence of the light incident on the cornea is -10.00 D when the spectacle lens is a -11.6-D lens worn 1.39 cm in front of the cornea. Hence, a contact lens correction for the same eye would have to have a dioptric power of -10.00 D.

III. VERGENCE—MEDIA OTHER THAN $n = 1$

Let us first consider a flat interface between media whose indices of refraction are n_1 and n_2. If plane waves in n_1 are incident on the interface, then the waves leaving the interface in n_2 will also be plane waves. The interface neither converges nor diverges the waves, although it will deviate the direction of travel according to Snell's law. The dioptric power of the interface is zero.

However, when converging or diverging waves are incident on the interface, then the curvature of the wave front does change as the light passes through the interface, which results in the well-known apparent depth effects. In order that the formalism of Eq. (2) be able to handle this situation, the definition of vergence is generalized: The vergence of light at a given position in a medium of index of refraction n is defined to be n times the curvature of the wave front at that position. Historically, such a generalization was introduced by the Swedish ophthalmologist Alvar Gullstrand (1862–1930) under the name "reduced vergence." At an interface, the radius of curvature of the incident wave front is identified as the object distance u, and the radius of curvature of the emerging wave front is identified as the image distance v. Therefore

$$u = n_1/U \tag{7}$$

and

$$v = n_2/V, \tag{8}$$

where U is the vergence of the incident light and V is the vergence of the emerging light.

In general, light incident on a spherical interface between media n_1 and n_2 will be converged or diverged by the interface. The curvature of the outgoing wave front can be related to the curvature of the incident wave front and the curvature of the interface by the use of the sagittal approximation.[12,13] The resulting equation in terms of the radius of curvature of the outgoing wave front, v, the radius of curvature of the incident wave front, u, and the radius of curvature of the interface, R, is

$$n_2/v = (n_2 - n_1)/R + n_1/u. \tag{9}$$

The sign convention is the same as in Sec. I.

Using Eqs. (7) and (8) and comparing Eq. (9) to Eq. (2), one sees that Eq. (9) has the same form as Eq. (2) and that the dioptric power P of the interface is

$$P = (n_2 - n_1)/R. \tag{10}$$

As an example, consider the light diffusely reflected by the iris of the human eye and traveling back through the aqueous humor and the cornea out into air. The iris is located approximately 3.6 mm behind the cornea. Assume that the cornea and the aqueous humor both have an index of 1.336. (The index of the cornea is actually closer to 1.376.) A typical radius of curvature of the cornea is 7.7 mm. From Eq. (10), the dioptric power of the cornea $P = (1.000 - 1.336)(1000 \text{ mm/m})(-7.7 \text{ mm})^{-1} = +43.6$ D. From Eq. (9), $U = -1336/3.6 = -371.1$ D. From Eq. (2), $V = +43.6 \text{ D} + (-371.1 \text{ D}) = -327.5$ D. The outgoing light is diverging and in air. From Eq. (10), the radius of curvature of the outgoing wave front (the image distance) is $v = 1000/(-327.5) = -3.05$ mm. In general, for a single spherical refracting interface, the magnification m is equal to U/V, or $m = -371.1/(-327.5) = +1.13$. Consequently, whenever you look at the beautiful brown eyes of either a male or female, you are seeing a virtual image that is slightly closer and slightly larger than the actual iris. (On a ray diagram, remember that the nodal ray goes through the center of curvature of the interface as Snell's law will easily show.)

If the light incident on the interface consists of plane waves ($U = 0$), then $f_2 = v$ and from Eqs. (2) and (8)

$$f_2 = n_2/P. \tag{11}$$

Similarly,

$$f_1 = -n_1/P. \tag{12}$$

A simple spherical lens consists of two interfaces separated by a certain thickness of glass or plastic. If the lens is considered thin, then the thickness is neglected and the vergence of the light leaving the first interface is equal to the vergence of the light incident on the second interface. The dioptric power of the thin lens is then just the sum of the dioptric powers of the two interfaces. Consider a glass (index 1.50) lens in air with the dioptric power of one side being +10 D (a convex side) and the dioptric power of the other side being −6 D (a concave side). The dioptric power of the lens is then +4 D, and the lens has a posterior focal length of 25 cm [Eq. (4)]. If this lens is now placed under water (index 1.33), the dioptric power will be reduced by a factor of $0.17/0.50 = 0.34$ [from Eq. (10)]. The dioptric power of the lens in the water is then +1.36 D, and from Eq. (11) the posterior focal length $f_2 = 133/1.36 = +97.8$ cm.

IV. CONCLUSION

The vergence approach reduces the algebra of geometric optics to the very basic operations of multiplication, division, addition, and subtraction. No common denominators and no manipulation of fractions are used, but even more important, the vergence approach provides a method of thinking in terms of what the light is doing at each position. The wave-front definition of vergence provides the student with a means of conceptualizing the processes, and it also crystallizes the meaning of many of the terms and phrases used in geometric optics. For example: "appears to be diverging from x" means the center of curvature of the actual diverging wave is at x; "the far point of the hyperope's eye is virtual and behind the eye at y" means that in order to have a sharp image on the retina of the hyperope the incident light must be converging, and the center of curvature of the actual incident wave front must be at y. The wave-front approach to vergence provides a clear representation of the fact that the vergence is different at each position as the light propagates through a uniform medium. This is important in understanding and computing the difference between spectacle lens power and a contact lens power.

The above-mentioned advantages are particularly useful for life science students. My observation, based on working with optometry students, is that the typical student exposed first to the vergence approach (complete with ray diagrams) understands the traditional approach as a by-product. On the other hand, the typical student exposed first to the traditional approach has a much harder time learning to think in terms of vergence. I think that physics courses designed for the life science student should treat geometric optics in terms of vergences.

The combined wave-front–vergence approach has one other advantage that has not been discussed in this paper. The student becomes familiar with the idea of light as a wave, and the concepts of interference and diffraction fall into place easier. Similarly, the student is better prepared for an introduction to optical transfer functions, which are now an integral part of courses that deal with information processing by the visual system.

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THE PREFIX MILLI

Millikan	a standard of achievement equal to 10^{-3} kan
Millipicture	a picture worth one word
Millihelen	beauty sufficient to launch one ship