# The complex form of the Fourier series

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In addition to the "standard" form of the Fourier series, there is a form using complex exponentials instead of the sine and cosine functions. This form is in fact easier to derive, since the integrations are simpler, and the process is also similar to the complex form of the Fourier integral.

## 1 Derivation of the complex form

Begin with a real periodic function $F(t)$ with period $T$:

$$F(t+T) = F(t). \tag{1}$$

We are going to write this as a series in complex exponentials

$$\begin{aligned} F(t) = a_0 + a_1 e^{i\omega t} + a_2 e^{2i\omega t} + \cdots + a_n e^{ni\omega t} + \cdots \\ + a_{-1} e^{-i\omega t} + a_{-2} e^{-2i\omega t} + \cdots + a_{-n} e^{-ni\omega t} + \cdots \end{aligned} \tag{2}$$

using both positive and negative indices, so we can write

$$F(t) = \sum_{n=-\infty}^{n=+\infty} a_n e^{in\omega t}. \tag{3}$$

Here $\omega = 2\pi/T$. Note that since we have assumed $F(t)$ is real, the complex coefficients on the right side above must add up to a real result. Now we will integrate over a period $T = 2\pi/\omega$ :

$$\int_0^{2\pi/\omega} F(t)\,dt = \int_0^{2\pi/\omega} \left( \sum_{n=-\infty}^{n=+\infty} a_n e^{in\omega t} \right) dt \tag{4}$$

and assuming we can interchange integration and summation:

$$= \sum_{n=-\infty}^{n=+\infty} a_n \int_0^{2\pi/\omega} e^{in\omega t}\,dt \tag{5}$$

Now consider the integral in the right-hand side, for various values of $n$.

$$\int_0^{2\pi/\omega} e^{in\omega t}\,dt = \frac{1}{i\omega n}(e^{2\pi i n} - 1) = 0 \qquad \text{for } n \neq 0, \text{ and} \tag{6}$$

$$= \int_0^{2\pi/\omega} dt = 2\pi/\omega = T \qquad \text{for } n = 0. \tag{7}$$

This gives us

$$\int_0^T F(t)\,dt = a_0 T, \tag{8}$$

$$a_0 = \frac{1}{T}\int_0^T F(t)\,dt. \tag{9}$$

which is the average value of $F$.

Now we will obtain the other coefficients. Start over again from

$$F(t) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\omega t}. \tag{10}$$

Note that for clarity further below, I am using a different label for the indices: $m$. This index is a "dummy," like an integration variable, so we may call it whatever we like. Multiply both sides of this by $e^{-in\omega t}$:

$$e^{-in\omega t}F(t) = e^{-in\omega t}\left(\sum_{m=-\infty}^{m=+\infty} a_m e^{im\omega t}\right), \tag{11}$$

and again integrate over a period, exchanging integration and summation. In essence, we are symbolically doing this for many particular $n$'s:

$$\int_0^{2\pi/\omega} e^{-in\omega t}F(t)\,dt = \sum_{m=-\infty}^{m=+\infty} \int_0^{2\pi/\omega} e^{i(m-n)\omega t}\,dt. \tag{12}$$

Now note that because the complex exponential is periodic,

$$\int_0^{2\pi/\omega} e^{i\omega(m-n)t}\,dt = 0 \qquad \text{except when } m = n \tag{13}$$

and $e^0 = 1$ when $m = n$. So we have

$$\int_0^{2\pi/\omega} e^{-in\omega t}F(t)\,dt = a_n T. \tag{14}$$

The coefficients of the series are thus

$$a_n = \frac{1}{T}\int_0^{2\pi/\omega} e^{-in\omega t}F(t)\,dt, \tag{15}$$

$$a_{-n} = \frac{1}{T}\int_0^{2\pi/\omega} e^{+in\omega t}F(t)\,dt. \tag{16}$$

We also see that

$$a_{-n} = (a_n)^*,$$

the negative index coefficients are the complex conjugates of the positive ones.

## 2 Back to the real form

To return to nonnegative indices and real sines and cosines is just a matter of rewriting now. Write the sum in eq. 3 in separate parts:

$$F(t) = \sum_{n=-\infty}^{n=-1} a_n e^{in\omega t} + a_0 + \sum_{n=1}^{n=+\infty} a_n e^{in\omega t} \tag{17}$$

$$= \sum_{n=\infty}^{n=1} a_{-n} e^{-in\omega t} + a_0 + \sum_{n=1}^{n=+\infty} a_n e^{in\omega t} \tag{18}$$

$$= a_0 + \sum_{n=1}^{n=\infty} (a_n e^{in\omega t} + a_{-n} e^{-in\omega t}) \tag{19}$$

Using Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$,

$$F(t) = a_0 + \sum_{n=1}^{n=\infty} (a_n + a_{-n})\cos n\omega t + \sum_{n=1}^{n=\infty} i(a_n - a_{-n})\sin n\omega t. \tag{20}$$

We now let

$$A_n = a_n + a_{-n}, \quad B_n = i(a_n - a_{-n}), \quad \frac{A_0}{2} = a_0, \tag{21}$$

and get

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{n=\infty} A_n \cos n\omega t + \sum_{n=1}^{n=\infty} B_n \sin n\omega t, \tag{22}$$

the usual real form of the Fourier series. We can work out the coefficient explicitly to see where all the constants come from:

$$A_n = a_n + a_{-n} = \frac{1}{T}\int_0^T F(t)(e^{-in\omega t} + e^{in\omega t}) \tag{23}$$

$$= \frac{2}{T}\int_0^T F(t)\cos n\omega t, \tag{24}$$

and

$$B_n = i(a_n - a_{-n}) = \frac{1}{T}\int_0^T F(t) i(e^{-in\omega t} - e^{in\omega t}) \tag{25}$$

$$= \frac{2}{T}\int_0^T F(t)\sin n\omega t. \tag{26}$$

Note that the $A_0/2$ term for the series (eq. 22) will be obtained by getting $A_0$ from the $n = 0$ coefficient of the cosine coefficients (24). The constant term in a Fourier series is always equal to the mean value of the function.

## 3 Magnitude-angle form

Yet another form of the Fourier series can be obtained from eq. 22 by writing

$$A_n \cos n\omega t + B_n \sin n\omega t = C_n \cos(n\omega t - \phi_n) \qquad = C_n \cos n\omega t \cos\phi_n + C_n \sin n\omega t \sin\phi_n. \tag{27}$$

Equate the coefficients of like terms

$$A_n = C_n \cos\phi_n, \quad B_n = C_n \sin\phi_n, \tag{28}$$

and so

$$C_n = \sqrt{A_n^2 + B_n^2}, \quad \phi_n = \arctan\tfrac{B_n}{A_n}. \tag{29}$$

The magnitude-angle form of the series is thus

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{n=\infty} C_n \cos(n\omega t - \phi_n) \tag{30}$$

or

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{n=\infty} C_n \sin(n\omega t + \tfrac{\pi}{2} - \phi_n). \tag{31}$$

The complex form of the Fourier series has many advantages over the real form. For example, integration and differentiation term-by-term is much easier with exponentials. The trigonometric functions and phase angles do not appear explicitly but are contained in the complex coefficients.

## 4 Reference

This material (with minor changes) is closely based on

*Applied Mathematics for Engineers and Physicists, Third ed.* by Pipes and Harvill, McGraw-Hill Publishers, 1970, Appendix C, section 22.