Derivation of optical dispersion for a single type of atomic oscillator.

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Plan

Figure out $\epsilon(\omega)$ from the relation

$$(\epsilon - \epsilon_0)\vec{E} = \vec{P}. \quad (1)$$

P depends on displacement of charges, so use classical model of forced oscillator to estimate this.

Then use Maxwell relation $n^2 = K_E = \epsilon/\epsilon_0$ to get $n^2(\omega)$

Driving force (electric field of wave)

$$F_E = q_e E(t) = q_e E_0 \cos \omega t. \tag{2}$$

Restoring force (electron orbital)

$$F = -k_E x \tag{3}$$

write in terms of ω_0:

$$\omega_0 = \sqrt{\frac{k_E}{m_e}}, \tag{4}$$

$$k_E = m_e \omega_0^2, \tag{5}$$

$$F = -m_e \omega_0^2 x, \tag{6}$$

Newton's second law:

$$q_e E_0 \cos \omega t - m_e \omega_0^2 x = m_e \frac{d^2 x}{dt^2}. \tag{7}$$

Trial solution:

$$x(t) = x_0 \cos \omega t. \tag{8}$$

Substitute and solve for x_0:

$$q_e E_0 \cos \omega t - m_e \omega_0^2 x_0 \cos \omega t = \ldots \tag{9}$$

$$q_e E_0 - m_e \omega_0^2 x_0 = -m_e \omega^2 x_0, \tag{10}$$

Rearrange, get

$$x_0 = \frac{q_e/m_e}{(\omega_0^2 - \omega^2)} E_0. \tag{11}$$

Substitute back into $x(t)$:

$$x(t) = \frac{q_e/m_e}{(\omega_0^2 - \omega^2)} E_0 \cos \omega t. \tag{12}$$

Polarization per volume:

$$P = q_e x N, \tag{13}$$

N is the number of polarizable atoms/volume

$$P = \frac{q_e^2 N E / m_e}{(\omega_0^2 - \omega^2)}. \tag{14}$$

Relation between P and ϵ:

$$(\epsilon - \epsilon_0) E = P$$

to get

$$\epsilon = \epsilon_0 + \frac{P(t)}{E(t)} = \epsilon_0 + \frac{q_e^2 N / m_e}{(\omega_0^2 - \omega^2)}, \tag{15}$$

Maxwell's relation:

$$n^2 = K_E = \epsilon/\epsilon_0$$

gives

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right). \tag{16}$$

Note that below a resonance $\omega < \omega_0$, so $n > 1$. Above: $\omega > \omega_0$, $n < 1$, $v_{\text{phase}} > c$.

At resonance, this diverges, but actually there is damping.