Power series

D. Craig, WTAMU

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A function can be represented by a series of powers times coefficents

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

with the coefficients determined by

$$c_n = \frac{1}{n!} \frac{d^n f}{dx^n}(x = 0),$$

for what is called an "expansion about $x = 0$."*
This is often written

$$f(x) = f(0) + x\frac{df}{dx}(x = 0) + \frac{x^2}{2}\frac{d^2 f}{dx^2}(x = 0) + \cdots$$

*The expansion about 0 is also known as a Maclaurin series, with Taylor the general case.

Expansion can also be made about any arbitrary $x = h$:

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \frac{d^n f}{dx^n}(x).$$

Keeping only the first two terms immediately leads to

$$f(x+h) - f(x) \approx h \frac{df}{dx}(x),$$
$$\frac{df}{dx} \approx \frac{f(x+h) - f(x)}{h}.$$

This should look familiar from introductory calculus, and also is the starting point for many numerical approximations.

“Classic” series

These are always worth remembering:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots ;$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots ;$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n;$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Radius of convergence and analyticity

Consider this carefully:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \text{with coefficients}$$
$$c_n = \frac{1}{n!}\frac{d^n f}{dx^n}(x = 0).$$

This suggests $f(x)$ is specified for all x when all its derivatives are known at $x = 0$. Intriguing, but not always true:

The series is infinite, it has a limited **radius of convergence**, and the function may not be **analytic** for all x.

Look at

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

A series converges (give a finite value) when all the terms go to zero as $n \to \infty$. For the above series, the terms are x^n, they only go to zero as $n \to \infty$ if $|x| < 1$.

A Taylor series only converges when

$$x < \text{radius of convergence},$$

which is a critical value.

If the function changes character in the region of interest, it may not be **analytic.** For example,

$$x(t) = \begin{cases} x_0 & \text{for } t \leq t_0, \\ x_0 + v(t - t_0) & \text{for } t > t_0. \end{cases}$$

This function is continuous,* but it is not analytic. What happens to the derivatives at $t = t_0$?

The full definition of analyticity uses concepts from complex analyis, which we see in 16.1 and 17.1. For the moment think of it as "having continuous derivatives."

*but only "piecewise continuous."

Practical note on approximations

It is often useful to use Taylor series to approximate a function at some point. One may wonder, what to do if the point one is interested in is outside the radius of convergence? Then use the expansion about $x = h$:

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \frac{d^n f}{dx^n}(x),$$

where h is in your region of interest.

Series are used to approximate special functions numerically, though usually not Taylor series, which typically require many terms for a good approximation.