Changing coordinate systems

Math Methods, D. Craig

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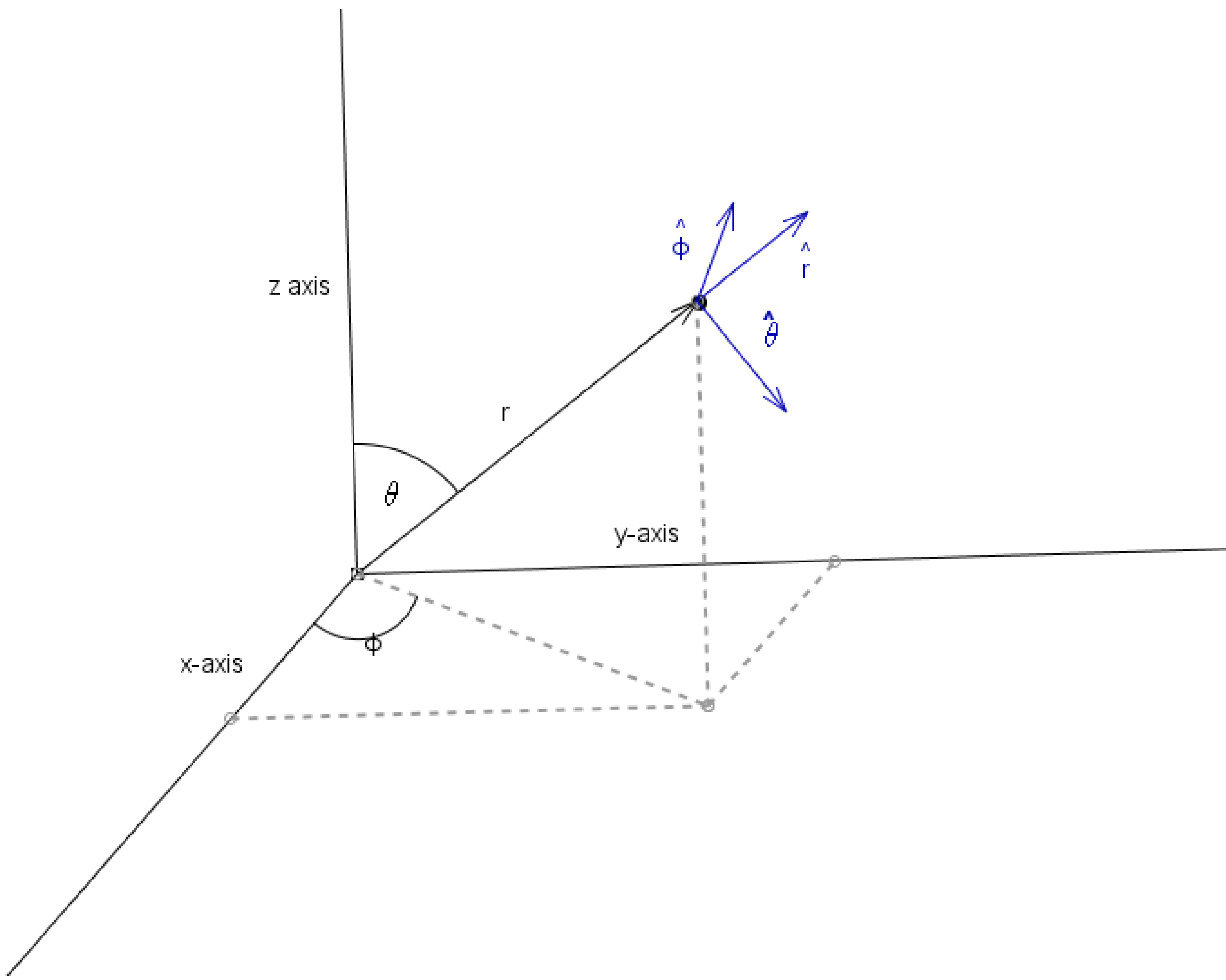

z axis
r
θ
y-axis
x-axis
ϕ
ϕ̂
r̂
θ̂

The relation between the unit vectors can be written in the symbolic form:

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix},$$

with $\mathbf{M}$ given by

$$\mathbf{M} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}.$$

This way of writing it is not completely legit, since we have “vectors of vectors” but it is a convenient shorthand for eqn. 4.13.

The coordinates (u_r, u_θ, u_ϕ) and (u_x, u_y, u_z) can be related by plugging the these expressions into $\vec{\mathbf{u}} = u_r\hat{\mathbf{r}} + u_\theta\hat{\theta} + u_\phi\hat{\phi}$:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix}$$

And we see that this is the transpose of the previous matrix:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \mathbf{M}^T \begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix}$$

Remember the transpose is to swap rows and columns: $M^T_{ij} = M_{ji}$.

Now we want the relation to run the other way, we want the spherical coordinates in terms of the cartesian coordinates. Multiply by the inverse $(\mathbf{M}^T)^{-1}$:

$$(\mathbf{M}^T)^{-1}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} = (\mathbf{M}^T)^{-1}\mathbf{M}^T\begin{pmatrix}u_r\\u_\theta\\u_\phi\end{pmatrix}$$

matrix times its inverse gives identity:

$$(\mathbf{M}^T)^{-1}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} = \mathbf{I}\begin{pmatrix}u_r\\u_\theta\\u_\phi\end{pmatrix}$$

which acts like the number 1:

$$(\mathbf{M}^T)^{-1}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} = \begin{pmatrix}u_r\\u_\theta\\u_\phi,\end{pmatrix}$$

So now we have what we want, but what is $(\mathbf{M}^T)^{-1}$?

All the columns and rows of **M** are orthogonal (check by dot products). So **M** is an **orthogonal matrix**, which means

$$\mathbf{M}^{-1} = \mathbf{M}^{T}$$

This can be verified by calculating $\mathbf{M}\mathbf{M}^{T}$ and $\mathbf{M}^{T}\mathbf{M}$ (prob 4.2d)

So $\mathbf{M}^{-1} = \mathbf{M}^{T}$, So

$$(\mathbf{M}^{T})^{-1} = (\mathbf{M}^{-1})^{-1} = \mathbf{M}$$

and so

$$\begin{pmatrix} u_r \\ u_\theta \\ u_\phi \end{pmatrix} = \mathbf{M} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}.$$

Why?

Consider

$$\vec{\mathbf{v}} = v_r \hat{\mathbf{r}} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$$

in spherical coordinates. If one takes derivatives to see what this is explicitly:

$$\vec{\mathbf{v}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r \sin\theta \dot{\phi}\hat{\phi}.$$

This because, for example: $\hat{\mathbf{r}} = \hat{\mathbf{r}}(\theta, \phi)$. The spherical unit vectors are dependent on the coordinates. The basis vectors change as a particle moves.

So this machinery is needed to convert coordinates. The acceleration is even more complex (see Problem 4.3e and equation 4.27.)

Volume integration

$$\int\int\int F\,dV = \int\int\int F(x, y, z)\,dx dy dz.$$

Volume spanned by three vectors:

$$\text{volume} = \det(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}) = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}.$$

Let $\theta \to \theta + d\theta$, then $\vec{\mathbf{r}}(r, \theta, \phi) \to \vec{\mathbf{r}}(r, \theta + d\theta, \phi)$ which gives a change

$$\vec{\mathbf{r}}(r, \theta + d\theta, \phi) - \vec{\mathbf{r}}(r, \theta, \phi) = \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\,d\theta.$$

This will be the infinitesimal vector along the θ direction. Similar reasoning applies to the others. So we get a little volume element:

$$dV = \det\left(\frac{\partial \vec{\mathbf{r}}}{\partial r}\,dr, \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\,d\theta, \frac{\partial \vec{\mathbf{r}}}{\partial \phi}\,d\phi\right)$$

The symbolic determinant is the $dV = J dr d\theta d\phi$, the volume element. It can be written as in problem 4.4b, equation 4.31. The usual shorthand notation for J is:

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}.$$

If you plug in the expression for $\vec{\mathbf{r}}$ and take many partials and work out the determinant for spherical coordinates, you find that.

$$J = r^2 \sin\theta,$$

So the spherical volume element is

$$dV = r^2 \sin\theta dr d\theta d\phi.$$

This can be done by geometric arguments (as you have probably seen before this) but the Jacobian handles more general coordinate systems.

Cylindrical coordinates

The text shows a clever set of substitutions for deriving equivalent results for cylindrical coordinates, based on the fact that they are the same at the equator of the sphere:

$$r = \sqrt{x^2 + y^2 + z^2} \to \sqrt{x^2 + y^2},$$
$$\theta \to \pi/2,$$
$$\hat{\theta} \to -\hat{\mathbf{z}},$$
$$r d\theta \to -dz.$$