Theorem of Gauss

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Gauss's law

$$\oint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} = \oint_V (\nabla \cdot \vec{\mathbf{v}}) dV.$$

"The closed surface integral of the flux is the volume integral of the divergence"

"What goes in must come out."

In one dimension

$$v_x(b) - v_x(a) = \int_a^b \frac{\partial v_x}{\partial x} dx$$

Spherically symmetric gravitational mass

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = g(r)\hat{\mathbf{r}}$$

The gravitational field satisfies

$$(\nabla \cdot \vec{\mathbf{g}}) = -4\pi G\rho$$

So

$$\oint_S \vec{\mathbf{g}} \cdot d\vec{\mathbf{S}} = -4\pi G \int_V \rho dV = -4\pi GM.$$

for a surface that completely encloses a mass M. Then doing the surface integral

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

Note that as in the electrostatics case, it is only the distribution of mass *inside the surface* that matters.

This implies that inside a hollow planet, the gravitional field vanishes, just like the electric field inside a hollow conductor.

Inside a sphere of constant density, outer radius R

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = -\frac{MGr}{R^3}\hat{\mathbf{r}}$$