Pre-class draft of the steps in section 8.3 TTVN Math Methods

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1 Introduction

This is a worked out set of the steps in section 8.3 of Snieder. This is a more compact form for reading than the slide format I usually use. You can print this out to follow the steps with a bit more commentary.

I may revise it a bit before presentation Wednesday, and I will probably recast it as slides for presentation on the screen, though I probably won't include all the steps.

As you follow this, try to visualize what the terms and integrals mean.

2 The acoustic representation theorem

The acoustic pressure field $p(\vec{r})$ satusfies the following PDE in frequency domain:

$$\nabla\left(\frac{1}{\rho}\nabla p\right)+\frac{\omega^2}{\kappa}p=f$$

$\rho(\vec{r})$ is the mass-density, ω is angular frequency, $\kappa(\vec{r})$ is compressibility. $f(\vec{r})$ is the source (driving) term, such as the pressure exerted by speaker in air, or a rock movement in an earthquake in the earth's crust. This is a driven wave equation, in a medium that may vary in its properties with position.

Consider two fields $p_1(\vec{r})$ and $p_2(\vec{r})$ that satisfy this, with two sources $f_1(\vec{r})$ and $f_2(\vec{r})$. (In all of the following $\vec{r}$ and $\vec{r}_0$ are position vectors denoting points in our medium.)

Multiply the PDE above for p_1 by p_2, and for p_2 by p_1, subtract them, and integrate over a volume of interest: [1]

$$\int_V\left[p_2\nabla\cdot\left(\frac{1}{\rho}\nabla p_1\right)-p_1\nabla\cdot\left(\frac{1}{\rho}\nabla p_2\right)\right]dV=\int_V(p_2f_1-p_1f_2)dV.$$

[1] Say, the Permian basin, or the room containing your stereo.

For a vector $\vec{\mathbf{v}}$ and a scalar field f, you can expand out the components to show that

$$\nabla \cdot (f\vec{\mathbf{v}}) = f(\nabla \cdot \vec{\mathbf{v}}) + \vec{\mathbf{v}} \cdot \nabla f.$$

Then

$$\nabla \cdot \left(\frac{1}{\rho} p_2 \nabla p_1\right) = p_2 \nabla \cdot \left(\frac{1}{\rho}\nabla p_1\right) + \frac{1}{\rho}(\nabla p_1 \cdot \nabla p_2),$$

rearranging

$$p_2 \nabla \cdot \left(\frac{1}{\rho}\nabla p_1\right) = \nabla \cdot \left(\frac{1}{\rho} p_2 \nabla p_1\right) - \frac{1}{\rho}(\nabla p_1 \cdot \nabla p_2),$$

and likewise for p_1, p_2 in the other order. This is the result for problem 8.3c Now put the two versions of this back in the expression

$$\int_V \left[p_2 \nabla \cdot \left(\frac{1}{\rho}\nabla p_1\right) - p_1 \nabla \cdot \left(\frac{1}{\rho}\nabla p_2\right)\right] \, dV$$

You will have two identical terms of $(1/\rho)(\nabla p_1 \cdot \nabla p_2)$, they will cancel. You are left with

$$\int_V \left[\nabla \cdot \left(\frac{1}{\rho} p_2 \nabla p_1\right) - \nabla \cdot \left(\frac{1}{\rho} p_1 \nabla p_2\right)\right] \, dV$$

This is the difference of two divergences, so by Gauss's Law, we turn this into a surface integral. Then the big integral expression we started with becomes:

$$\oint_S \frac{1}{\rho}(p_2 \nabla p_1 - p_1 \nabla p_2) \cdot d\vec{\mathbf{S}} = \int_V (p_2 f_1 - p_1 f_2) \, dV.$$

To see what this means, we restrict to the special case of a point source (see text). Let the source term $f_2 \neq 0$ only in a tiny neighborhood around $\vec{\mathbf{r}}_0$. [2]

$$\oint_S \frac{1}{\rho}(p2 \nabla p_1 - p_1 \nabla p_2) \cdot d\vec{\mathbf{S}} = \int_V (p_2 f_1) \, dV - p(\vec{\mathbf{r}}_0)$$

The wavefield p_2 is generated by this "point" source, is called the Green's function—it is response of system at one point to a localized disturbance elsewhere:

$$p_2 \to G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)$$

So

$$\oint_S \frac{1}{\rho}(G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)\nabla p_1 - p_1 \nabla G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)) \cdot d\vec{\mathbf{S}} = \int_V (p_2 f_1) \, dV - p(\vec{\mathbf{r}}_0)$$

Let's have no other sources within the volume: $f_1 \to 0$, and we drop the subscript on p_1, taking p as our "response".

$$\oint_S \frac{1}{\rho}(G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)\nabla p - p \nabla G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)) \cdot d\vec{\mathbf{S}} = -p(\vec{\mathbf{r}}_0)$$

[2] Or be a delta function source: Ch. 14.

and rearrange the minus signs to get the desired result:

$$p(\vec{\mathbf{r}}_0) = \oint_S \frac{1}{\rho}(p(\vec{\mathbf{r}})\nabla G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0) - G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)\nabla p(\vec{\mathbf{r}})) \cdot d\vec{\mathbf{S}}$$

What does this equation mean? $\vec{\mathbf{r}}_0$ is an arbitray point inside the volume. $G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)$ is the wavefield that would appear at $\vec{\mathbf{r}}$ due to a unit point source at $\vec{\mathbf{r}}_0$. $p(\vec{\mathbf{r}})$ is the wavefield say, detected at $\vec{\mathbf{r}}$.

If we know $G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0), p(\vec{\mathbf{r}}), \nabla p(\vec{\mathbf{r}})$ for points on a boundary, we can get $p(\vec{\mathbf{r}}_0)$ for **any arbitrary** $\vec{\mathbf{r}}_0$ **within the volume** bounded by it.

The catch—what is $G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)$? Often this can be produced by successive approximation based on preliminary estimates, and partial data, but the devil is in the details.