8.3: Acoustic representation theorem

Math Methods, D. Craig

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Acoustic representation theorem

- Application of Gauss Theorem to wave problem.

- First look at Green's functions— "response" of system to localized disturbance

- Representation theorems—determine inside behavior by boundary behavior

- leads to Huygen's principle for acoustic waves.

- Inverse problems

The acoustic pressure field $p(\vec{\mathbf{r}})$ satusfies the following PDE in frequency domain:

$$\nabla\left(\frac{1}{\rho}\nabla p\right)+\frac{\omega^2}{\kappa}p=f$$

$\rho(\vec{\mathbf{r}})$ is the mass-density, ω is angular frequency, $\kappa(\vec{\mathbf{r}})$ is compressibility. $f(\vec{\mathbf{r}})$ is the source (driving) term.

Consider two fields $p_1(\vec{\mathbf{r}})$ and $p_2(\vec{\mathbf{r}})$ that satisfy this, with two sources $f_1(\vec{\mathbf{r}})$ and $f_2(\vec{\mathbf{r}})$.

Multiply each field by the other and subtract the resulting expressions, then look at volume integral:

$$\int_V \left[p_2 \nabla \cdot \left(\frac{1}{\rho} \nabla p_1 \right) - p_1 \nabla \cdot \left(\frac{1}{\rho} \nabla p_2 \right) \right] dV$$
$$= \int_V (p_2 f_1 - p_1 f_2) dV.$$

use

$$\nabla \cdot (f\vec{\mathbf{v}}) = f(\nabla \cdot \vec{\mathbf{v}}) + \vec{\mathbf{v}} \cdot \nabla f.$$

This will make the left term inside the LHS integral above:

$$p_2 \nabla \cdot \left(\frac{1}{\rho} \nabla p_1 \right) = \nabla \cdot \left(\frac{1}{\rho} p_2 \nabla p_1 \right) - \frac{1}{\rho} (\nabla p_1 \cdot \nabla p_2),$$

and similarly for the other order for p_1, p_2. This is sort of like integration by parts.

Put all this in and cancel identical terms. The left integral will be

$$\int_V \left[\nabla \cdot \left(\frac{1}{\rho} p_2 \nabla p_1\right) - \nabla \cdot \left(\frac{1}{\rho} p_1 \nabla p_2\right)\right] \, dV$$

A difference of two integrated divergences. We use Gauss Theorem

$$\oint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} = \int_V (\nabla \cdot \vec{\mathbf{v}}) \, dV,$$

to turn this into a surface integral, so that

$$\oint_S \frac{1}{\rho}(p_2 \nabla p_1 - p_1 \nabla p_2) \cdot d\vec{\mathbf{S}} = \int_V (p_2 f_1 - p_1 f_2) \, dV.$$

Now p_2 is driven by f_2, and p_1 by f_1 We are going to specialize to a case where the driving f_2 is non-zero only at a point $\vec{\mathbf{r}}_0$, so p_2 is the response of the medium to this special input.

$$p_2(\vec{\mathbf{r}}) = G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)$$

We will suppose we know this for our medium, for any $\vec{\mathbf{r}}_0$. This is the Green's function for our medium.

p_1 will be the overall solution we are seeking, supposing we know G. We will suppose there are no sources for the p_1 wavefield, so $f_1 = 0$ everywhere.

Now we make a lot of substitutions.

The non-obvious step in getting problem e is that, if p_2 is localized at a point $\vec{\mathbf{r}}_0$ then

$$\int_V p_1 f_2 \, dV = p_1(\vec{\mathbf{r}}_0)$$

This is because we have $f_2 = 0$ everywhere except at $\vec{\mathbf{r}}_0$.

This is equivalent to using a *Dirac delta function* source

$$f_2 = \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$$

Where the delta function is **defined** as having the property

$$\int_V g(\vec{\mathbf{r}})\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) \, dV = g(\vec{\mathbf{r}}_0)$$

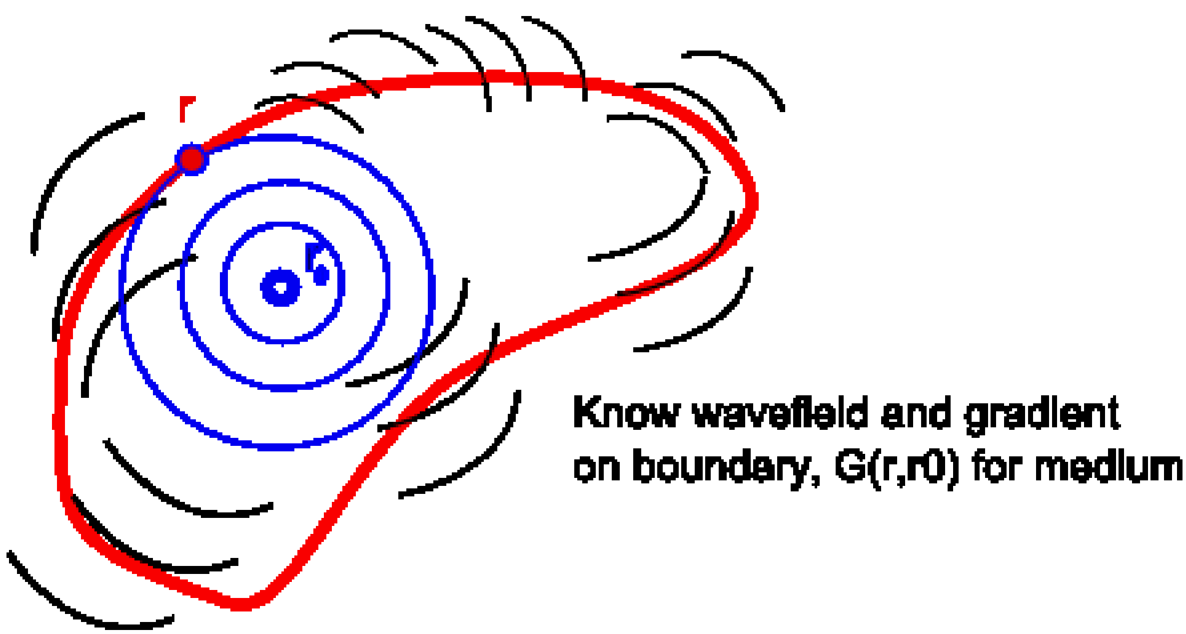
for $\vec{\mathbf{r}}_0$ inside the volume. This is a way of handling "point" functions or "spikes" in integrals.

When we put all this together we get

$$p(\vec{\mathbf{r}}_0) = \oint_S \frac{1}{\rho}\left(p(\vec{\mathbf{r}})\nabla G(\vec{\mathbf{r}},\vec{\mathbf{r}}_0) - G(\vec{\mathbf{r}},\vec{\mathbf{r}}_0)\nabla p(\vec{\mathbf{r}})\right)\cdot d\vec{\mathbf{S}}$$

What does this equation mean?

If we know $G(\vec{\mathbf{r}},\vec{\mathbf{r}}_0), p(\vec{\mathbf{r}}), \nabla p(\vec{\mathbf{r}})$ for points on a boundary, we can get $p(\vec{\mathbf{r}}_0)$ for **any arbitrary $\vec{\mathbf{r}}_0$ within the volume** bounded by it.



But what is G?

The catch—what is $G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_0)$? Often this can be produced by successive approximation based on preliminary estimates, and partial data, but the devil is in the details.

This is usually done interatively: estimate G from known properties, model the medium based on known inputs, compare results to inputs, improve G.

Once you have a good estimate of G, you know everything about wave propagation in the medium.

This is known as an "inverse problem."