9.6 Wingtip vortices

Math Methods, D. Craig

2007–02–28

How do airplanes fly?

. . . is a far more subtle question than you may realize.

Section 9.6 is a neat example but the physical explanation is a bit sketchy. One thing that is not clearly stated is the *Kutta-Zhukovsky theorem:**

Lift = airspeed·circulation·air density·wingspan

For a good general reference on all things flight-related, see

`http://www.av8n.com/how/`

by John S. Denker. The drawings there are fantastic, much better than your text.

*Zhukovsky is a Russian name, you may see it transliterated many ways, including Joukowsky!

Problem a: Is the circulation $\oint_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}}$ positive or negative in fig 9.10 for contour C?

Think about $\vec{\mathbf{v}} \cdot d\vec{\mathbf{r}}$ as you go around C. On the top, the airflow is both faster and opposite the sense of C, so the top makes a large negative contribution. The bottom of the flow is slow, but aligned with the sense of C, this gives a small positive contribution.

Conclusion: the circulation is *negative:*

$$\oint_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} < 0. \tag{1}$$

Problem b: See figure 9.11. Just apply Stoke's theorem to the circulation:

$$\oint_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} = \int_S (\nabla \times \vec{\mathbf{v}}) \cdot d\vec{\mathbf{S}} \qquad (2)$$

For a surface S enclosing the wingtip. The vorticity is just

$$\vec{\omega} = \nabla \times \vec{\mathbf{v}}, \qquad (3)$$

so Stoke's theorem becomes

$$\oint_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} = \int_S \vec{\omega} \cdot d\vec{\mathbf{S}}. \qquad (4)$$

which we sought to show. The circulation around a contour is the sum of the vorticity through a surface bounded by that contour.

The vorticity is $\nabla \times \vec{\mathbf{v}}$. Remember that ∇ involves spatial derivatives. Where will the flow be changing most rapidly in space? Near the wingtips. Wings produce lift when the circulation along $C \neq 0$. Wingtip vortices are associated with lift.

Problem c: does the wingtip vortex in fig 9.11 rotate clockwise (A) or counterclockwise (B)?

Look carefully at the figure. Note $\hat{\mathbf{n}}$ pointing out from S. We know that

$$\oint_C \vec{\mathbf{v}} \cdot d\vec{\mathbf{r}} < 0.$$

This implies that

$$\int_S \vec{\omega} \cdot d\vec{\mathbf{S}} = \int_S (\nabla \times \vec{\mathbf{v}}) \cdot d\vec{\mathbf{S}} < 0. \qquad (5)$$

In terms of $\hat{\mathbf{n}}$:

$$\int_S (\nabla \times \vec{\mathbf{v}}) \cdot \hat{\mathbf{n}}\, dS < 0, \qquad (6)$$

So, for the vortex near the tip where the contribution to the integral is strongest,

$$(\nabla \times \vec{\mathbf{v}}) \cdot \hat{\mathbf{n}} = \vec{\omega} \cdot \hat{\mathbf{n}} < 0, \qquad (7)$$

which implies the vorticity vector $\vec{\omega}$ points *opposite* to $\hat{\mathbf{n}}$. By the right-hand-rule, this means the vortex is clockwise in the picture, choice **A.**

Now does this make physical sense?

The opposite wing will have the opposite sense for everything, so a counterclockwise vortex. See the figure at:

`http://www.av8n.com/how/htm/airfoils.html#fig-trailing`

The vortices produce descending air behind the plane. The wings exert forces downward on this air, changing its momentum to have a downward componen. By Newton's Third law, this produces an upward force on the wings: the lift. When Newton's laws are satisfied, we feel pretty happy.

I heartily recommend the "See how it flies". The author is both a pilot and a physicist.