Laplace's equation, harmonic averaging

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The variational analysis of section 10.2 can be extended to to a 2–D soap film, as in section 10.3, most of which we'll leave for homework.

For 1–D we had

$$\delta L[h] = -\int_a^b \frac{d^2h}{dx^2}\epsilon(x)\,dx, \tag{1}$$

which vanishes for *any* $\epsilon(x)$ if $L[h]$ is stationary.

For a surface:

$$\delta S[h] = -\int\int \epsilon(x,y)\nabla^2 h(x,y)\ dxdy, \tag{2}$$

where h is the "height" of the surface in Cartesian coordinates, and this variation must vanish for any $\epsilon(x,y)$ that is zero at the boundary.

In both cases this means the other term in the integrand (involving h) must vanish as well.

For the stationary arclength, we found no curvature:

$$\frac{d^2h}{dx^2} = 0, \tag{3}$$

which implies $h(x)$ is a straight line.

For the stationary surface area we find

$$\nabla^2 h(x, y) = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0. \tag{4}$$

This does *not* require a flat surface, because a positive curvature in say the x-direction, can be canceled by a negative curvature in the y-direction (or many other possibilities).

For example suppose

$$h(x, y) = x^2 - y^2, \tag{5}$$

(prob 10.3h) it is easy to see that $\nabla^2 h = 0$ in this case, which is a "saddle-shaped" surface.

Now think about a stationary point:

$$\frac{\partial h}{\partial x} = 0, \qquad \frac{\partial h}{\partial y} = 0. \tag{6}$$

This can't be a minimum or maximum because $\nabla^2 h = 0$ implies that if it is a minimum in one direction, it must be a maximum in a perpendicular direction, or equivalently that $\frac{\partial^2 h}{\partial x^2}$ and $\frac{\partial^2 h}{\partial y^2}$ cannot have the same sign.

So, *a function that satisfies* $\nabla^2 h = 0$ *can only have a maximum or minimum at the edge of the domain on which it is defined.* (p. 124)

Averaging integrals (10.8)

Laplace equation in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}=0. \tag{7}$$

Integrate over a circle of radius R:

$$\iint_S g(r,\phi)dA=\int_0^R\int_0^{2\pi} g(r,\phi)rd\phi dr \tag{8}$$

This gives

$$\int_0^R\int_0^{2\pi}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 f}{\partial \phi^2}\right]d\phi dr=0. \tag{9}$$

The last term in integrand will be the same at $0, 2\pi$ in the definite integral, so it makes no contribution:

$$\int_0^R\int_0^{2\pi}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right]d\phi dr=0. \tag{10}$$

This can be written:

$$\int_0^R \left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^{2\pi} f d\phi\right)\right] dr = 0. \tag{11}$$

Define

$$\bar{f}(r) \equiv \frac{1}{2\pi}\int_0^{2\pi} f(r,\phi)d\phi. \tag{12}$$

So

$$\int_0^R \left[\frac{\partial}{\partial r}\left(r\frac{\partial \bar{f}}{\partial r}\right)\right] dr = 0, \tag{13}$$

$$\left[r\frac{\partial \bar{f}}{\partial r}\right]_{r=0}^{r=R} = 0. \tag{14}$$

This holds for any R, so it's a constant:

$$r\frac{\partial \bar{f}}{\partial r} = C, \tag{15}$$

which integrates to

$$\bar{f}(r) = C \ln(r) + A \tag{16}$$

but we can't have an infinity, $\bar{f}$ is finite as $r \to 0$, so $C = 0$ and $f(r = 0) = \bar{f} = A$, so

$$f(r = 0) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \phi) d\phi. \tag{17}$$

The value of the harmonic function f at the origin is given by the average over a circle of any radius centered on the origin. Since the origin is just a point, this holds true for any point: average on a circle around it and get the value at the center.

This puts strong constraints on the behavior of the function, and since harmonic functions are so common in physics through $\nabla^2 f = 0$ this gives many nice properties to lots of functions of physical interest.