Some residue integration examples

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2007-03-24

Here are a couple of examples of contour integration using residues and a contour in the upper half-plane. They are exercises from *Complex variables, harmonic and analytic functions,* by Francis J. Flanigan.

1 First example

Integrate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}. \tag{1}$$

Use the complex function

$$f(z) = \frac{1}{z^2 + z + 1}. \tag{2}$$

We will use a contour that goes from $-R$ to R on the real axis, and then close it with a semicircular contour in the upper half of the complex plane, so $C = C_{\text{real}} + C_R$. We let $R \to \infty$ to get the whole real axis.

First we have to make sure that along the semicircular C_R, the contribution of the integral vanishes. We use Flanigan's "lemma" which is that the integral over C_R vanishes if

$$\lim_{R=|z|\to\infty} zf(z) = 0.$$

So consider

$$\lim_{R=|z|\to\infty} \frac{z}{z^2 + z + 1} = \lim_{|z|\to\infty} \frac{1}{z + 1 + 1/z} = 0, \tag{3}$$

which satisfies the lemma.[1] So the integral along the real axis will be given by the residue of the pole in the upper half-plane.

The denominator of $f(z)$ has roots at

$$z = \frac{-1 \pm i\sqrt{3}}{2}$$

[1]This is based on the ML-inequality or "estimation lemma", see Wikipedia, etc.

So

$$f(z) = \frac{1}{[z - \frac{1}{2}(-1 - i\sqrt{3})][z - \frac{1}{2}(-1 + i\sqrt{3})]}, \tag{4}$$

call $z_0 = \frac{1}{2}(-1 + i\sqrt{3})$, this is the location of the pole in the upper half-plane, which will be enclosed by the contour. The pole is simple so we can find the residue using

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{5}$$

The residue is

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z - z_0)}{[z - \frac{1}{2}(-1 - i\sqrt{3})](z - z_0)} \tag{6}$$

$$= \left[\frac{1}{2}(-1 + i\sqrt{3} + 1 + i\sqrt{3}) \right]^{-1} \tag{7}$$

$$= \frac{1}{i\sqrt{3}} \tag{8}$$

So

$$\int_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}, \tag{9}$$

$$= 2\pi i \,\text{Res}(f, z_0), \tag{10}$$

$$= 2\pi i \left(\frac{1}{i\sqrt{3}} \right), \tag{11}$$

$$= \frac{2\pi}{\sqrt{3}}. \tag{12}$$

2 Slightly more complicated example.

Integrate

$$\int_{-\infty}^{\infty} \frac{x - 2^{1/3}}{x^3 - 2}\, dx, \tag{13}$$

where $2^{1/3}$ is the real cube root of 2. (Remember there are n nth roots of a number, some of which are complex.)

We are going to use the same contour as before. First, check that the integral will vanish along C_R:

$$\lim_{|z| \to \infty} z f(z) = \lim_{|z| \to \infty} \frac{z(z - 2^{1/3})}{z^3 - 2} \tag{14}$$

$$= \lim_{|z| \to \infty} \left[\frac{z^2}{z^3 - 2} - \frac{2^{1/3} z}{z^3 - 2} \right] = 0, \tag{15}$$

so we don't have to worry about C_R. Now what are the poles of

$$f(z) = \frac{(z - 2^{1/3})}{z^3 - 2}? \tag{16}$$

They will be the zeros of $z^3 - 2$, i. e. the three cube roots of 2, which are

$$2^{1/3},\ 2^{1/3}e^{i2\pi/3},\ 2^{1/3}e^{i4\pi/3}.$$

The numerator of $f(z)$ will eliminate the first, which is on the real axis. The second will be in the upper half-plane, so it is inside C, the last is in the lower half-plane.[2]

Now follow the procedure:

$$f(z) = \frac{(z - 2^{1/3})}{(z - 2^{1/3})(z - 2^{1/3}e^{i2\pi/3})(z - 2^{1/3}e^{i4\pi/3})} \tag{17}$$

$$= \frac{1}{(z - 2^{1/3}e^{i2\pi/3})(z - 2^{1/3}e^{i4\pi/3})}. \tag{18}$$

Our enclosed pole is at $z_0 = 2^{1/3}e^{i2\pi/3}$, so we can get

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{z - z_0}{(z - z_0)(z - 2^{1/3}e^{i4\pi/3})}, \tag{19}$$

$$= \lim_{z \to z_0} \frac{1}{(z - 2^{1/3}e^{i4\pi/3})}, \tag{20}$$

$$= \left[2^{1/3}(e^{i2\pi/3} - e^{i4\pi/3})\right]^{-1} \tag{21}$$

working out the real and imaginary parts of the exponentials, get

$$\mathrm{Res}(f, z_0) = \frac{1}{2^{1/3}3^{1/2}i}. \tag{22}$$

Now we put this all together and

$$\int_{-\infty}^{\infty} \frac{x - 2^{1/3}}{x^3 - 2}\, dx = 2\pi i \,\mathrm{Res}(f, z_0), \tag{23}$$

$$= \frac{2\pi}{2^{1/3}3^{1/2}}. \tag{24}$$

This looks like it would be very difficult to obtain using real variable methods.

[2] If this is confusing, review the properties of complex roots. In any case, it is a good idea to sketch a diagram of the root locations.