Draft on Tensors

D. Craig

2007–02–11

1 Tensors as multilinear forms

Consider a function that takes a vector argument and returns a scalar.

$$\mathbf{t}(\vec{v}) = \text{ a scalar}$$

Let it be *linear* in the vector argument:

$$\mathbf{t}(\vec{u} + \vec{v}) = \mathbf{t}(\vec{u}) + \mathbf{t}(\vec{v}) \tag{1}$$
$$\mathbf{t}(\alpha\vec{v}) = \alpha\mathbf{t}(\vec{v}) \tag{2}$$

where α is an arbitrary scalar. This is called a *linear form.* Now consider a scalar function of *two* vectors, that is linear in both arguments

$$\mathbf{T}(\vec{u} + \vec{r}, \vec{v}) = \mathbf{T}(\vec{u}, \vec{v}) + \mathbf{T}(\vec{r}, \vec{v}) \tag{3}$$
$$\mathbf{T}(\alpha\vec{u}, \vec{v}) = \alpha\mathbf{T}(\vec{u}, \vec{v}), \tag{4}$$
$$\mathbf{T}(\vec{u}, \vec{v} + \vec{w}) = \mathbf{T}(\vec{u}, \vec{v}) + \mathbf{T}(\vec{u}, \vec{w}) \tag{5}$$
$$\mathbf{T}(\vec{u}, \beta\vec{v}) = \beta\mathbf{T}(\vec{u}, \vec{v}). \tag{6}$$

This is a *bilinear form.*

A *multilinear form* is a scalar function of several vectors:

$$\mathbf{Q}(\vec{u}, \vec{v}, \ldots \vec{z})$$

which is linear in each of its arguments $(\vec{u}, \vec{v}, \ldots \vec{z})$.

As you might guess, we are going to call these things *tensors.* If we consider only cartesian space, this is about all we have to do: a tensor takes of rank N takes N vectors and returns a scalar, if it takes $N - 1$ it returns a vector. But for a number of reasons, it is better to generalize, and let vectors be a form of tensor. This requires the introduction of dual vectors.

2 Dual vectors: 1-forms

Take an arbitrary linear form. We will denote it $\tilde{p}$. When it is supplied with a vector $\vec{v}$, we get a scalar $\tilde{p}(\vec{v})$. Take another one $\tilde{q}$, and we can define

$$\tilde{s} = \tilde{p} + \tilde{q}, \tag{7}$$

$$\tilde{t} = \alpha\tilde{p} \tag{8}$$

to be the forms whose values for $\vec{v}$ are

$$\tilde{s}(\vec{v}) = \tilde{p}(\vec{v}) + \tilde{q}(\vec{v}), \tag{9}$$

$$\tilde{t}(\vec{v}) = \alpha\tilde{p}(\vec{v}). \tag{10}$$

With this, we see that the set of all linear forms of one vector argument satisfy the requirements for a vector space. We will call these *one-forms.* They form a *dual* or *adjoint* vector space.

Consider the dot product in Cartesian space, it is a bilinear form:

$$\vec{u} \cdot \vec{v} = \mathbf{g}(\vec{u}, \vec{v}). \tag{11}$$

For a particular fixed vector $\vec{a}$

$$\vec{a} \cdot (\quad) = \mathbf{g}(\vec{a}, \quad) \tag{12}$$

we see that $\mathrm{g}(\vec{a}, \quad)$ is a one-form, because it is "waiting" for another vector argument, if given one it will produce a scalar. We can use this to construct a dual vector for any vector $\vec{v}$, by using the idea

$$\tilde{v} = \mathbf{g}(v, \quad)$$

where g is whatever function yields our dot product for our space.

(Comments about row and column vectors and dual spaces here?

3 Tensors that take vector arguments

Call a tensor that takes one vector argument a type $\binom{0}{1}$, and one that takes two vectors type $\binom{0}{2}$.

3.1 The tensor product

The simplest type of these $\binom{0}{2}$ tensors is formed as follows: $\tilde{p} \otimes \tilde{q}$ is the tensor which produces the number $(\tilde{p}\vec{u})(\tilde{q}\vec{v})$ when supplied with $\vec{u}$ and $\vec{v}$ as arguments, in other words, just the product of the numbers produced by the one-forms. This operation is *not* commutative:

$$\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p},$$

because

$$(\tilde{p} \otimes \tilde{q})(\vec{\mathbf{u}}, \vec{\mathbf{v}}) = (\tilde{p}(\vec{u}))(\tilde{q}(\vec{v}))$$
$$(\tilde{q} \otimes \tilde{p})(\vec{\mathbf{u}}, \vec{\mathbf{v}}) = (\tilde{q}(\vec{u}))(\tilde{p}(\vec{v})).$$

The most general $\binom{0}{2}$ tensor is not a simple tensor product, but it can always be represented as a sum of such products.

So far, I have avoided components or matrices in these notes. A note about the Kronecker product of matrices migt be appropriate here.

4 References

As of 2007–01–11 This is based mostly on

- Akivis and Goldberg
- Schutz, *A first course in general relativity,* Ch. 3.1–3.5